

Quantum critical properties of the Bose-Fermi Kondo model in a large- N limit

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Studies of non-Fermi liquid properties in heavy fermions have led to the current interest in the Bose-Fermi Kondo model. Here we use a dynamical large- N approach to analyze an $SU(N) \times SU(\kappa N)$ generalization of the model. We establish the existence in this limit of an unstable fixed point when the bosonic bath has a sub-ohmic spectrum ($|\omega|^{1-\epsilon} \text{sgn} \omega$, with $0 < \epsilon < 1$). At the quantum critical point, the Kondo scale vanishes and the local spin susceptibility (which is finite on the Kondo side for $\kappa < 1$) diverges. We also find an ω/T scaling for an extended range (15 decades) of ω/T . This scaling violates (for $\epsilon \geq 1/2$) the expectation of a naive mapping to certain classical models in an extra dimension; it reflects the inherent quantum nature of the critical point.

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Heavy fermion systems show unusual quantum critical properties, which have motivated the interest in how the Kondo effect is destroyed near a magnetic quantum phase transition[1, 2, 3, 4]. One theoretical approach studies Kondo lattice systems through a self-consistent Bose-Fermi Kondo model[1]. Here, a local magnetic moment not only interacts antiferromagnetically with the spins of a conduction electron bath, it also is coupled to a dissipative bosonic bath. (For work in a related context, see Ref. 5.) The conduction electrons alone would lead to a Kondo singlet for the ground state and spin- $\frac{1}{2}$ charge- e Kondo resonances in the excitation spectrum[6]. The bosonic bath – characterizing the fluctuating magnetic field provided by spin fluctuations – competes with the Kondo effect; a sufficiently strong bosonic coupling destroys the Kondo effect. A quantum critical point (QCP) arises, where the electronic excitations are of non-Fermi liquid nature. When such a Kondo-destroying criticality is embedded into the criticality of a magnetic ordering, the critical theory becomes distinct from the standard paramagnon theory[1]. This picture has been called locally quantum critical. Direct evidence for the destruction of the Kondo effect has recently emerged in the Hall measurements of a heavy fermion metal[7].

Previous studies of the Bose-Fermi Kondo model are based on an ϵ -expansion renormalization group (RG) analysis[1, 8, 9, 10, 11, 12] [the definition of ϵ appears in Eq. (2) below; $\epsilon > 0$ corresponds to sub-ohmic bosonic bath]. This approach has been successful, partly due to the fact that the linear in ϵ contribution to the anomalous dimension of the local spin operator turns out to be exact to all orders in ϵ [10, 11, 12]. In spite of this, the approach is after all perturbative in ϵ ; moreover, it is capable of calculating only a limited number of physical quantities. (The model lacks conformal invariance, making boundary conformal field theory inapplicable.) For the case with an Ising spin-anisotropy, a self-consistent version of the model has been studied[13] by a Quantum Monte-Carlo method[14]. With spin rotational invariance, the only approach that has been used, beyond the ϵ -expansion,

is the condensed slave-boson mean field theory[15]; here, the QCP is trivial: the effect of the Kondo interaction disappears completely as soon as the static slave-boson amplitude vanishes. A dynamical method is necessary to capture the quantum critical properties.

In this paper, we study the model using a dynamical large- N method[16, 17, 18]. The spin symmetry is taken to be $SU(N)$ and the number of conduction electron channels $M \equiv \kappa N$. The large- N limit is expected to capture the quantum-critical behavior of the physical cases (see below). This limit, with purely Kondo coupling to fermions was extensively studied in Refs. [16, 17] and with purely bosonic coupling in Ref. [18] (see also [9, 10]).

Our primary results are three-fold. First, we establish the existence of a non-trivial QCP even in the large- N limit. Second, we show that the QCP is non-Gaussian not only for small ϵ but also for $\epsilon \geq 1/2$. This demonstrates the inherently quantum nature of the QCP: standard mapping of a quantum critical point to a classical critical point at extra dimensions[23] would imply that $\epsilon = 1/2$ is the upper critical dimension (*c.f.* the classical spin chains with $1/r^{2-\epsilon}$ interactions[19, 20]); the difference arises from the Berry phase of the quantum spin. Third, we determine the critical exponents and amplitudes and the universal scaling functions near the QCP, as well as the properties on the approach towards the bosonic fixed point. We will restrict to $\kappa \equiv M/N < 1$ for which the spin susceptibility of the multi-channel Kondo fixed point is finite[16, 17, 21].

The Hamiltonian for the model is

$$\begin{aligned} \mathcal{H}_{\text{MBFK}} = & (J_K/N) \sum_{\alpha} \mathbf{S} \cdot \mathbf{s}_{\alpha} + \sum_{p, \alpha, \sigma} E_p c_{p\alpha\sigma}^{\dagger} c_{p\alpha\sigma} \\ & + (g/\sqrt{N}) \mathbf{S} \cdot \boldsymbol{\Phi} + \sum_p w_p \boldsymbol{\Phi}_p^{\dagger} \cdot \boldsymbol{\Phi}_p. \end{aligned} \quad (1)$$

Here, a local moment \mathbf{S} interacts with fermions $c_{p\alpha\sigma}$ and bosons $\boldsymbol{\Phi}_p$. The spin and channel indices are $\sigma = 1, \dots, N$ and $\alpha = 1, \dots, M$, respectively, and $\boldsymbol{\Phi} \equiv \sum_p (\boldsymbol{\Phi}_p + \boldsymbol{\Phi}_{-p}^{\dagger})$ contains $N^2 - 1$ components. We will first consider a flat conduction electron density of

states and a subohmic bosonic spectrum. Denoting $\mathcal{G}_0 = -\langle T_\tau c_{\sigma\alpha}(\tau) c_{\sigma\alpha}^\dagger(0) \rangle_0$ and $\mathcal{G}_\Phi = \langle T_\tau \Phi(\tau) \Phi^\dagger(0) \rangle_0$ for each component, we have $N_0(\omega) \equiv -\frac{1}{\pi} \text{Im} \mathcal{G}_0(\omega + i0^+) = N_0$ ($|\omega| < D = 1/2N_0$) and $A_\Phi(\omega) \equiv \frac{1}{\pi} \text{Im} \mathcal{G}_\Phi(\omega + i0^+) = \sum_p [\delta(\omega - w_p) - \delta(\omega + w_p)]$ being

$$A_\Phi(\omega) = [K_0^2/\Gamma(2-\epsilon)]|\omega|^{1-\epsilon} \text{sgn}(\omega), \quad (2)$$

for $|\omega| < \Lambda \equiv 1/\tau_0$ (Γ is the gamma function). We consider the conduction electrons in the fundamental representation of the $\text{SU}(N) \times \text{SU}(M)$ group, and the local moment in an antisymmetric representation whose Young tableaux is a single column of $N/2$ boxes. We can then write S in terms of pseudo-fermions, $S_{\sigma\sigma'} = f_\sigma^\dagger f_{\sigma'} - \delta_{\sigma,\sigma'}/2$, with the constraint $\sum_{\sigma=1}^N f_\sigma^\dagger f_\sigma = N/2$, which is enforced by a Lagrange multiplier $i\mu$. The Kondo coupling, $(J_K/N) \sum_{\sigma\sigma'} (f_\sigma^\dagger f_{\sigma'} - \delta_{\sigma,\sigma'}/2) c_{\alpha\sigma'}^\dagger c_{\alpha\sigma}$, for each channel α , will be decoupled into a $B_\alpha^\dagger \sum_\sigma c_{\alpha\sigma}^\dagger f_\sigma / \sqrt{N}$ interaction using a dynamical Hubbard-Stratonovich field $B_\alpha(\tau)$. It is convenient to consider g and J_K as independently varying, since the universal properties only depend on the ratio g/T_K^0 .

Renormalization group analysis: We will first establish that a non-trivial QCP survives the large- N limit. We have generalized the RG equations of Ref. 11 to arbitrary N and M . In the large- N limit, the RG beta functions, perturbative only in ϵ , become

$$\begin{aligned} \beta(\bar{g}) &= -2\bar{g}(\epsilon/2 - \bar{g} + \bar{g}^2 - \kappa\bar{j}^2), \\ \beta(\bar{j}) &= -\bar{j}(\bar{j} - \kappa\bar{j}^2 - \bar{g} + \bar{g}^2), \end{aligned} \quad (3)$$

where, $\bar{g} \equiv (K_0g)^2$, and $\bar{j} \equiv N_0J_K$. There is an unstable fixed point at $(\bar{g}^*, \bar{j}^*) = (\epsilon/2 + (1-\kappa)\epsilon^2/4, \epsilon/2)$. Here, $\chi(\tau) \sim 1/|\tau|^\eta$ and the anomalous dimension $\eta = \epsilon$.

Saddle-point equations: Our primary objective is to study the saddle-point equations of the large- N limit, which have the following form (cf. Fig. 1):

$$\begin{aligned} G_B^{-1}(i\omega_n) &= 1/J_K - \Sigma_B(i\omega_n); \quad \Sigma_B(\tau) = -\mathcal{G}_0(\tau)G_f(-\tau) \\ G_f^{-1}(i\omega_n) &= i\omega_n - \lambda - \Sigma_f(i\omega_n); \\ \Sigma_f(\tau) &= \kappa\mathcal{G}_0(\tau)G_B(\tau) + g^2G_f(\tau)\mathcal{G}_\Phi(\tau), \end{aligned} \quad (4)$$

together with a constraint equation,

$$G_f(\tau = 0^-) = (1/\beta) \sum_{i\omega_n} G_f(i\omega_n) e^{i\omega_n 0^+} = 1/2. \quad (5)$$

The dynamical spin susceptibility is

$$\chi(\tau) = -G_f(\tau)G_f(-\tau). \quad (6)$$

Here, λ is the saddle point value of $i\mu$, $G_B(\tau) = \langle T_\tau B_\alpha(\tau) B_\alpha^\dagger(0) \rangle$ and $G_f(\tau) = -\langle T_\tau f_\sigma(\tau) f_\sigma^\dagger(0) \rangle$.

The first and second terms on the RHS of the last line of Eq. (4) reflect the Kondo and bosonic couplings, respectively. They capture the competition between the

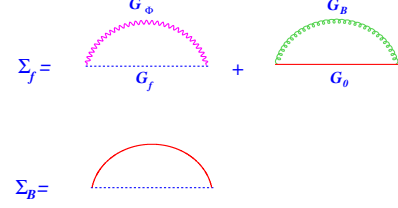


FIG. 1: The large- N Feynman diagrams for the self-energies.

two types of interactions. To proceed, we seek for scale-invariant solutions of the following form at $T = 0$,

$$G_f(\tau) = A_1/|\tau|^{\alpha_1} \text{sgn}\tau, \quad G_B(\tau) = B_1/|\tau|^{\beta_1}, \quad (7)$$

for the asymptotic regime $\tau \gg \tau_0$. Using these, the saddle-point equations become

$$\begin{aligned} G_f^{-1}(\omega) &= \frac{\omega^{1-\alpha_1}}{A_1 C_{\alpha_1-1}} = \omega + \kappa N_0 B_1 C_{\beta_1} \omega^{\beta_1} \\ &\quad - (K_0 g)^2 A_1 C_{1-\epsilon+\alpha_1} \omega^{1-\epsilon+\alpha_1} \\ G_B^{-1}(\omega) &= \frac{\omega^{1-\beta_1}}{B_1 \beta_1 C_{\beta_1}} \\ &= \frac{1}{J_K} - \Sigma_B(0) + N_0 A_1 (\alpha_1 + 1) C_{\alpha_1+1} \omega^{\alpha_1} \end{aligned} \quad (8)$$

where $C_{x-1} \equiv \frac{\pi}{\Gamma(x)} \frac{\exp[i\pi(2-x)/2]}{\sin[\pi(2-x)/2]}$. We have used $\lambda + \Sigma_f(0) = 0$, which follows from Eq. (5). We have also assumed $0 < \alpha_1, \beta_1 < 1$, which turns out to be valid in most cases; exceptions will be specified below.

Three solutions arise depending on the competition between the Kondo and bosonic coupling terms. A dominating Kondo or bosonic term leads to the Kondo or bosonic phase, respectively. When the two terms are of the same order, the critical fixed point emerges.

Multichannel Kondo and bosonic fixed points: When $\beta_1 < 1 - \epsilon + \alpha_1$, the Kondo coupling dominates over the bosonic coupling on the RHS of Eq. (8). The leading order solution is then identical to that of the model with only a Kondo coupling[16]. The leading exponents are $\alpha_1 = 1/(1+\kappa)$ and $\beta_1 = \kappa/(1+\kappa)$.

In the opposite case, with $\beta_1 > 1 - \epsilon + \alpha_1$, the bosonic term dominates over the Kondo term. Matching the dynamical parts of Eq. (8) leads to

$$\alpha_1 = \epsilon/2; \quad A_1^2 = \frac{2-\epsilon}{4\pi(K_0g)^2} \tan \frac{\pi\epsilon}{4}. \quad (10)$$

Together with Eq. (6), they combine to yield the following result for the dynamical spin susceptibility

$$\chi(\tau) = \left(\frac{2-\epsilon}{4\pi} \tan \frac{\pi\epsilon}{4} \right) \frac{1}{(K_0g)^2} \frac{1}{|\tau|^\epsilon}. \quad (11)$$

The exponent agrees with that of Refs. 8, 9, 10, 11, 12, 18.

If J_K is strictly 0, $G_B(\omega)$ vanishes. For small but finite J_K , Eq. (9), which (more precisely, the term in between

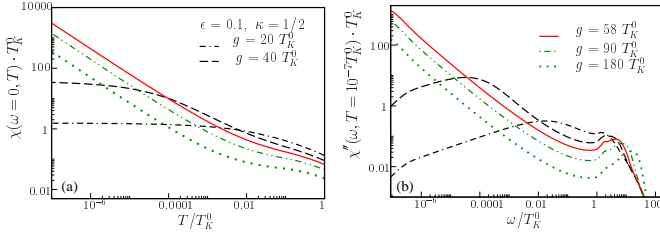


FIG. 2: Static and dynamical local susceptibilities for $\epsilon = 0.1$. The red solid line corresponds to the quantum critical point, $g = g_c$.

the two equalities) comes with the assumption $\beta_1 < 1$, cannot be satisfied. A solution does however exist for $\beta_1 > 1$, in which case $G_B(\omega)$ no longer diverges. It now follows from $G_B^{-1}(\omega) \sim \text{const} + \omega^{\beta_1-1}$ that

$$\beta_1 = 1 + \epsilon/2. \quad (12)$$

The non-divergence of G_B reflects the irrelevant nature of the Kondo interaction at the bosonic fixed point.

Critical fixed point: The Kondo and bosonic terms are of the same order when $\beta_1 = 1 - \epsilon + \alpha_1$. In this case, matching the dynamical parts leads to

$$\alpha_1 = \epsilon/2; \quad \beta_1 = 1 - \epsilon/2; \quad (13)$$

$$A_1^2 = \frac{2 - (1 + \kappa)\epsilon}{4\pi(K_0 g_c)^2} \tan \frac{\pi\epsilon}{4}; \quad B_1 = -\frac{\epsilon}{4\pi N_0 A_1} \tan \frac{\pi\epsilon}{4}. \quad (14)$$

The critical bosonic coupling g_c needs to be numerically determined by matching the static part: $1/J_K - \Sigma_B(0) = 0$. The resulting dynamical spin susceptibility is:

$$\chi(\tau) = \left(\frac{2 - (1 + \kappa)\epsilon}{4\pi} \tan \frac{\pi\epsilon}{4} \right) \frac{1}{(K_0 g_c)^2} \frac{1}{|\tau|^\epsilon}. \quad (15)$$

We make two observations at this point. First, for infinitesimal ϵ , our result agrees not only with the result of the ϵ -expansion for the large-N model [*c.f.* the subsection containing Eq. (3)] but also that of the ϵ -expansion for the the physically relevant SU(2) single-channel model[11, 12]. This implies that the large-N limit captures the quantum critical behavior of the physical systems, even though it yields a multichannel behavior on the Kondo side. This is reminiscent of the effects of spin anisotropies in the single-channel models: though the bosonic fixed point with Ising anisotropy (whose local susceptibility contains a finite Curie constant[11]) is very different from its counterpart with SU(2) symmetry, η is the same for the QCPs of the two cases[11]. Second, Eq. (15) goes beyond the ϵ -expansion result described earlier in the sense that it is valid beyond infinitesimal ϵ .

We have also determined the exponents of the subleading terms for G_f and G_B , finding $\alpha_2 = \epsilon$ and $\beta_2 = 1$.

We next consider approaching the QCP from the Kondo side. Defining T^* to be the crossover scale where the spectral function $A_f(\omega)$ changes from $A_f(\omega) \sim$

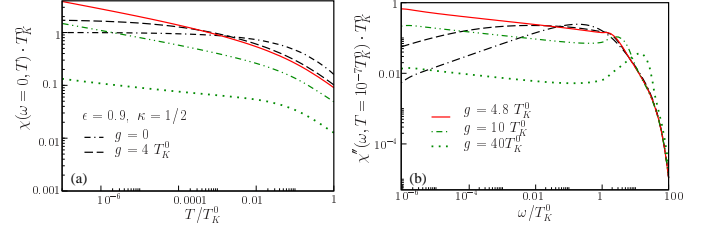


FIG. 3: Static and dynamical susceptibilities for $\epsilon = 0.9$.

$\frac{1}{\omega^{\kappa/(1+\kappa)}}$ for $|\omega| \ll T^*$ to $A_f(\omega) \sim \frac{1}{\omega^{1-\epsilon/2}}$ for $|\omega| \gg T^*$, and inserting the ansatz into $1/J_K = \Sigma_B(\omega = 0) = \int_0^\beta d\tau G_0(\tau) G_f(\tau)$, we find $T^* \sim (g_c - g)^{2/\epsilon}$. Correspondingly, the susceptibility diverges as g approaches g_c :

$$\chi(T, \omega = 0, g \lesssim g_c) \sim (g_c - g)^{-\gamma}, \quad \gamma = 2(1 - \epsilon)/\epsilon. \quad (16)$$

Our results so far represent an essentially complete analytical solution at zero temperature. The solution at finite temperatures is considerably more difficult to obtain. In the following, we resort to numerical studies.

Numerical results: For low-energy analysis, we find it important to solve the integral eqs. (8,9) in *real frequencies* and using a *logarithmic discretization* [supplemented by linearly-spaced points; typically 250 points in total for the frequency range $(-10, 10)$]. The convergence is determined in terms of $A_B(\omega)$ and $A_f(\omega)$: for each, the sum over the entire frequency range of the relative difference between two consecutive iterations is less than 10^{-5} . We choose $\kappa = 1/2$ and $N_0(\omega) = (1/\pi)\exp(-\omega^2/\pi)$. The nominal bare Kondo scale is $T_K^0 \equiv (1/N_0)\exp(-1/N_0 J_K) \approx 0.06$, for fixed $J_K = 0.8$. The bosonic bath spectral function [*cf.* Eq. (2)] is cut off smoothly at $\Lambda \approx 0.05$. K_0^2 is fixed at $\Gamma(2 - \epsilon)$.

In Fig. 2a), we show the static local spin susceptibility $\chi(T)$ as a function of temperature for $\epsilon = 0.1$. For vanishing and relatively small bosonic coupling g , $\chi(T)$ saturates to a finite value in the zero-temperature limit – which characterizes the Kondo phase. At a critical coupling[22] of $g/T_K^0 \approx 56$, $\chi(T)$ diverges in a power-law fashion. Beyond this threshold coupling, $\chi(T)$ continues to have a power-law divergence. Its amplitude, on the other hand, decreases with increasing g . These features are not restricted to small ϵ ; similar results are given in Fig. 3a), for $\epsilon = 0.9$. They are consistent with the analytical results: Eq. (16) states that $\chi(T = 0)$ should diverge as g approaches g_c from below; for $g > g_c$, on the other hand, Eq. (11) implies that the susceptibility amplitude should indeed decrease as g increases.

The quantum phase transition can also be seen in the dynamics. In Fig. 2b), we show the imaginary part of the dynamical local spin susceptibility, $\chi''(\omega)$, at the lowest studied temperature ($T = 10^{-7} T_K^0$) and for $\epsilon = 0.1$. On the Kondo side, it vanishes in the zero-frequency limit with an exponent close to 0.33, consistent with the analytical result $2\alpha_1 - 1 = 1/3$. At g_c , it displays a power-law

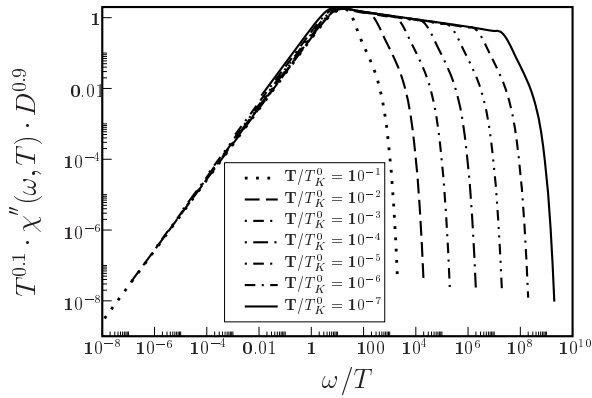


FIG. 4: ω/T scaling of the dynamical spin susceptibility at $g = g_c$, for $\epsilon = 0.9$ and $\kappa = 1/2$. The scaling exponent is 0.1

divergence, with an exponent close to 0.9. On the bosonic side, the exponent remains the same while the amplitude decreases as g increases. Both exponents agree with the analytical result of $1 - 2\alpha_1 = 1 - \epsilon = 0.9$. Similar results for $\epsilon = 0.9$ are shown in Fig. 3b).

We now address the finite frequency and temperature behavior at the QCP in some detail. Fig. 4 plots the dynamical spin susceptibility for $\epsilon = 0.9$, clearly manifesting an ω/T scaling. Remarkably, the scaling covers an overall 15 decades of ω/T (from 10^{-8} to about 10^7)! For each temperature, the result falls on the scaling curve until an ω of the order T_K^0 . Similar behavior is observed for other values of ϵ in the range $0 < \epsilon < 1$. The ω/T scaling reflects the interacting nature of the critical fixed point[23]: there is no energy scale other than T ; the relaxation rate, originating from some relevant coupling, has to be linear in T .

The small but visible deviation from a complete collapse in the range $\omega, T < T_K^0$ reflects the influence of subleading contributions. For $\epsilon = 0.9$, a similar degree of ω/T scaling is also seen in the B - and f -spectral functions $A_B(\omega, T)$ and $A_f(\omega, T)$. The imaginary part of the f -self-energy, $\Sigma_f''(\omega, T)$, on the other hand, starts to deviate from ω/T scaling at a smaller frequency, reflecting a stronger effect of the subleading terms. For $\epsilon = 0.1$, $A_B(\omega, T)$ and $\Sigma_f''(\omega, T)$ are less influenced by the subleading terms than $A_f(\omega, T)$ and $\chi''(\omega, T)$.

Our work serves as a basis for studies of the spin-isotropic Kondo lattice systems: by establishing the first non-perturbative approach to the Kondo-destroying QCP of the Bose-Fermi Kondo model, we are in the position to study the lattice problem in as systematic a way as the Quantum Monte-Carlo study of the Kondo lattice

with Ising anisotropy[13]. Our results are also relevant in two other contexts. First, some generalized version of the Bose-Fermi Kondo model may capture the physics of impurities in high T_c superconductors[24, 25]. Results similar to what we report here should shed light on the dynamical properties of such systems, which have not yet been systematically addressed. Second, the competition between Kondo and paramagnon couplings also appears in the context of disorder in nearly magnetic metals[26].

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